On the Global Calculus in Local Cohomology in BRST Theory

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Abstract. We show that the cohomology groups of the horizontal (total) differential on horizontal (local) exterior forms on the infinite-order jet manifold of an affine bundle coincide with the De Rham cohomology groups of the base manifold. This prevents one from the topological obstruction to definition of global descent equations in BRST theory on an arbitrary affine bundle.

1 Introduction

Let $Y \to X$ be a smooth fibre bundle of some classical field model. We study cohomology of exterior forms on the infinite-order jet space $J^\infty Y$ of $Y \to X$. This cohomology plays an important role in the field-antifield BRST formalism for constructing the descent equations [1-4].

In the framework of this BRST formalism, one considers the so-called horizontal complex

$$0 \to \mathbb{R} \to \mathcal{O}^0_\infty \xrightarrow{d_H} \mathcal{O}^{0,1}_\infty \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathcal{O}^{0,n}_\infty, \tag{1}$$

where $\mathcal{O}^{0,*}_\infty$ is a subalgebra of horizontal (semibasic) exterior forms on $J^\infty Y$, and d_H is the horizontal (total) differential. Extended to the jet space of ghosts and antifields, these forms and their cohomology are called local forms and local cohomology. Given the BRST operator $\mathbf{s}$, one defines the total BRST operator $\mathbf{s} + d_H$ and examines the BRST cohomology modulo d_H.

It should be emphasized that, the above mentioned BRST formalism is formulated on a contractible fibre bundle $Y = \mathbb{R}^{n+m} \to \mathbb{R}^n$. Of course, this is not the generic case of gauge theory and its outcomes to topological field models and anomalies. The key point is that, in this case, the horizontal complex (1) is exact. This fact called the algebraic Poincaré lemma is crucial for constructing the (local) descent equations in BRST theory. To write global descent equations in a non-trivial topological context, one should study (global)

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cohomology of the complex (1). The question is that there are (at least) two classes of exterior forms on the infinite-order jet space $J^\infty Y$, and both of them fail to be differential forms on $J^\infty Y$ in a rigorous sense because $J^\infty Y$ is not a Banach manifold [5-7].
Recall that the infinite-order jet space of a smooth fibre bundle $Y \to X$ is defined as a projective limit $(J^\infty Y, \pi^\infty_j)$ of the surjective inverse system

$$X \xleftarrow{\pi} Y \xleftarrow{\pi^1_0} \cdots \longleftarrow J^{r-1}Y \xleftarrow{\pi^r_{r-1}} J^r Y \longleftarrow \cdots \tag{2}$$

of finite-order jet manifolds $J^r Y$ [5-8]. Provided with the projective limit topology, $J^\infty Y$ is a paracompact Fréchet (but not Banach) manifold [5, 6, 9]. Its paracompactness will be an essential tool for cohomological calculations below. Given a bundle coordinate chart $(\pi^{-1}(U_X); x^\lambda, y^i)$ on the fibre bundle $Y \to X$, we have the coordinate chart $((\pi^\infty)^{-1}(U_X); x^\lambda, y^i_\Lambda)$, $0 \leq |\Lambda|$, on $J^\infty Y$, together with the transition functions

$$y'^i_{\lambda+\Lambda} = \frac{\partial x^\mu}{\partial x'^\lambda} d_\mu y'^i_\Lambda, \tag{3}$$

where $\Lambda = (\lambda_k ... \lambda_1)$, $|\Lambda| = k$, is a multi-index, $\lambda + \Lambda$ is the multi-index $(\lambda\lambda_k \dots \lambda_1)$ and d_λ are the total derivatives

$$d_\lambda = \partial_\lambda + \sum_{|\Lambda|=0} y^i_{\lambda+\Lambda} \partial^\Lambda_i.$$

The differential calculus on $J^\infty Y$ can be introduced as operations on the $\mathbb{R}$-ring $\mathcal{Q}^0_\infty$ of locally pull-back functions on $J^\infty Y$. A real function f on $J^\infty Y$ is called so if, for each point $q \in J^\infty Y$, there is a neighbourhood U_q such that $f|_{U_q}$ is the pull-back of a smooth function on some finite-order jet manifold $J^k Y$ with respect to the surjection π^∞_k. It should be emphasized that the paracompact space $J^\infty Y$ admits the partition of unity performed by elements of $\mathcal{Q}^0_\infty$ [5, 6]. The difficulty lies in the geometric interpretation of derivations of the $\mathbb{R}$-ring $\mathcal{Q}^0_\infty$ as vector fields on the Fréchet manifold $J^\infty Y$ and their dual as differential forms on $J^\infty Y$ [7].
Therefore, one usually considers the subring $\mathcal{O}^0_\infty$ of the ring $\mathcal{Q}^0_\infty$ which consists of the pull-back onto $J^\infty Y$ of smooth functions on finite-order jet spaces. The Lie algebra of derivations of $\mathcal{O}^0_\infty$ is isomorphic to the projective limit onto $J^\infty Y$ of the Lie algebras of projectable vector fields on finite-order jet manifolds. The associated algebra of differential forms is introduced as the direct limit $(\mathcal{O}^*_\infty, \pi^{\infty*}_k)$ of the direct system

$$\mathcal{O}^*(X) \xrightarrow{\pi^*} \mathcal{O}^*(Y) \xrightarrow{\pi^1_0{}^*} \mathcal{O}^*_1 \xrightarrow{\pi^2_1{}^*} \cdots \xrightarrow{\pi^r_{r-1}{}^*} \mathcal{O}^*_r \longrightarrow \cdots \tag{4}$$

of differential $\mathbb{R}$-algebras $\mathcal{O}^*_r$ of exterior forms on finite-order jet manifolds $J^r Y$. This direct limit exists in the category of $\mathbb{R}$-modules, and the direct limits of the familiar

operations on exterior forms make $\mathcal{O}^*_\infty$ a differential exterior $\mathbb{R}$-algebra. This algebra consists of all exterior forms on finite-order jet manifolds modulo the pull-back identification. Therefore, one usually thinks of elements of $\mathcal{O}^*_\infty$ as being the pull-back onto $J^\infty Y$ of exterior forms on finite-order jet manifolds. Being restricted to a coordinate chart $(\pi^\infty)^{-1}(U_X)$ on $J^\infty Y$, elements of $\mathcal{O}^*_\infty$ can be written in the familiar coordinate form, where basic forms $\{dx^\lambda\}$ and contact 1-forms $\{\theta^i_\Lambda = dy^i_\Lambda - y^i_{\lambda+\Lambda}dx^\lambda\}$ provide the local generators of the algebra $\mathcal{O}^*_\infty$. There is the canonical splitting of the space of m-forms

$$\mathcal{O}^m_\infty = \mathcal{O}^{0,m}_\infty \oplus \mathcal{O}^{1,m-1}_\infty \oplus \ldots \oplus \mathcal{O}^{m,0}_\infty$$

into spaces $\mathcal{O}^{k,m-k}_\infty$ of k-contact forms. Accordingly, the exterior differential on $\mathcal{O}^*_\infty$ is decomposed into the sum $d = d_H + d_V$ of horizontal and vertical differentials

$$\begin{array}{lll} d_H : \mathcal{O}^{k,s}_\infty \to \mathcal{O}^{k,s+1}_\infty, & d_H(\phi) = dx^\lambda \wedge d_\lambda(\phi), & \phi \in \mathcal{O}^*_\infty, \\ d_V : \mathcal{O}^{k,s}_\infty \to \mathcal{O}^{k+1,s}_\infty, & d_V(\phi) = \theta^i_\Lambda \wedge \partial^\Lambda_i \phi, & \end{array}$$

which obey the nilpotency rule

$$d_H \circ d_H = 0, \qquad d_V \circ d_V = 0, \qquad d_V \circ d_H + d_H \circ d_V = 0. \tag{5}$$

In studying the algebra $\mathcal{O}^*_\infty$ of pull-back exterior forms on $J^\infty Y$, the key point of lies in the fact that the infinite-order De Rham complex of these forms

$$0 \to \mathbb{R} \to \mathcal{O}^0_\infty \xrightarrow{d} \mathcal{O}^1_\infty \xrightarrow{d} \cdots \tag{6}$$

is the direct limit of the De Rham complexes of exterior forms on finite-order jet manifolds. Then, as was repeatedly proved, the cohomology groups $H^*(\mathcal{O}^*_\infty)$ of the complex (6) are equal to the De Rham cohomology groups $H^*(Y)$ of the fibre bundle Y [5, 6]. This fact enables one to say something on the topological obstruction to the exactness of the (infinite-order) variational complex in the calculus of variations in field theory [6,7,10-12]. At the same time, the d_H-cohomology of the horizontal complex (1) of pull-back exterior forms on $J^\infty Y$ remains unknown. The local exactness of this complex only has been repeatedly proved (see, e.g., [13, 14]). If a fibre bundle $Y \to X$ admits a global section, there is also a monomorphism of the De Rham cohomology groups $H^*(X)$ of the base X to the cohomology groups of the complex (1) [12].

Here, we show that the problem of cohomology of the horizontal complex has a comprehensive solution by enlarging the algebra $\mathcal{O}^*_\infty$ to the algebra $\mathcal{Q}^*_\infty$ of the above mentioned locally pull-back exterior forms on $J^\infty Y$. We introduce these forms in an algebraic way as global sections of the sheaf $\mathfrak{Q}^*_\infty$ of differential algebras on $J^\infty Y$ which is the direct limit of the direct system

$$\mathfrak{O}^*_X \xrightarrow{\pi^*} \mathfrak{O}^*_Y \xrightarrow{\pi^1_0{}^*} \mathfrak{O}^*_1 \xrightarrow{\pi^2_1{}^*} \cdots \xrightarrow{\pi^r_{r-1}{}^*} \mathfrak{O}^*_r \longrightarrow \cdots \tag{7}$$

of sheaves of exterior forms on finite-order jet manifolds J^rY. As a consequence, we have the exact sequence of sheaves

$$0 \to \mathbb{R} \to \mathfrak{Q}^0_\infty \xrightarrow{d_H} \mathfrak{Q}^{0,1}_\infty \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathfrak{Q}^{0,n}_\infty \tag{8}$$

of horizontal forms on $J^\infty Y$ and the corresponding complex of $\mathcal{Q}^0_\infty$-modules of their global sections

$$0 \to \mathbb{R} \to \mathcal{Q}^0_\infty \xrightarrow{d_H} \mathcal{Q}^{0,1}_\infty \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathcal{Q}^{0,n}_\infty. \tag{9}$$

Since $J^\infty Y$ is paracompact and admits a partition of unity by elements of $\mathcal{Q}^0_\infty$, all sheaves $\mathfrak{Q}^{0,m}_\infty$ in the exact sequence (8) are fine and, consequently, acyclic. Therefore, the well-known theorem on a resolution of a sheaf [15] can be applied in order to obtain the cohomology groups of the horizontal complex (9).

From the physical viewpoint, an extension of the class of exterior forms to $\mathcal{Q}^*_\infty$ enables us to concern effective field theories whose Lagrangians involve derivatives of arbitrary high order [16]

Here, we study d-, d_V- and d_H-cohomology of the horizontal complex (9) on the infinite-order jet space $J^\infty Y$ of an affine bundle $Y \to X$. Note that affine bundles provide a standard framework in quantum field theory because almost all existent quantization schemes deal with linear and affine quantities. Moreover, the De Rham cohomology groups of an affine bundle $Y \to X$ are equal to those of its base X. Therefore, as we will see, the obstruction to the exactness of the horizontal complex (9) lies only in exterior forms on X. Since the BRST operator $\mathbf{s}$ eliminate these forms, the global descent equations can be constructed though their right-hand sides are not equal to zero.

Moreover, we can restrict our consideration to vector bundles $Y \to X$ without loss of generality as follows. Let $Y \to X$ be a smooth affine bundle modelled over a smooth vector bundle $\overline{Y} \to X$. A glance at the transformation law (3) shows that $J^\infty Y \to X$ is an affine topological bundle modelled on the vector bundle $J^\infty\overline{Y} \to X$. This affine bundle admits a global section $J^\infty s$ which is the infinite-order jet prolongation of a global section s of $Y \to X$. With $J^\infty s$, we have a homeomorphism

$$\widehat{s}_\infty : J^\infty Y \ni q \mapsto q - (J^\infty s)(\pi^\infty(q)) \in J^\infty \overline{Y}$$

of the topological spaces $J^\infty Y$ and $J^\infty\overline{Y}$, together with an exterior algebra isomorphism $\widehat{s}^*_\infty : \overline{\mathcal{O}}^*_\infty \to \mathcal{O}^*_\infty$. Moreover, it is readily observed that the pull-back morphism $\widehat{s}^*_\infty$ commutes with the differentials d, d_V and d_H. Therefore, the differential algebras $\mathcal{Q}^*_\infty$ and $\overline{\mathcal{Q}}^*_\infty$ have the same d-, d_V- and d_H-cohomology.

Given a smooth vector bundle $Y \to X$, we will show the following.

(i) The De Rham cohomology groups of the differential algebra $\mathcal{Q}^*_\infty$ are isomorphic to those of the base X.

(ii) Its d_V-cohomology groups are trivial.

(iii) The d_H-cohomology groups of contact elements of the algebra $\mathcal{Q}^*_\infty$ are trivial.

(iv) The d_H-cohomology groups of its horizontal elements (i.e., cohomology of the horizontal complex (9)) coincide with the De Rham cohomology groups of the base X.

Note that the results (i) and (ii) are also true for the differential algebra $\mathcal{O}^*_\infty$. The result (iii) takes place for an arbitrary smooth fibre bundle $Y \to X$, and recovers that in Refs. [10, 11], obtained by means of the Mayer-Vietoris sequence.

2 Differential algebra $\mathcal{Q}^*_\infty$

Throughout the paper, smooth manifolds are real, finite-dimensional, Hausdorff, paracompact, and connected.

Given the surjective inverse system (2), we have the direct system (7) of ringed spaces $(J^kY, \mathfrak{O}^*_k)$ whose structure sheaves $\mathfrak{O}^*_k$ are sheaves of differential $\mathbb{R}$-algebras of exterior forms on finite-order jet manifolds J^kY, and ${\pi^r_{r-1}}^*$ are the pull-back morphisms. Throughout, we follow the terminology of Ref. [15] where by a sheaf is meant a sheaf bundle. The direct system (7) admits a direct limit $\mathfrak{Q}^*_\infty$ which is a sheaf of differential exterior $\mathbb{R}$-algebras on the infinite-order jet space $J^\infty Y$. This direct limit exists in the category of sheaves of $\mathbb{R}$-modules, and the direct limits of the familiar operations on exterior forms provide $\mathfrak{Q}^*_\infty$ with a differential exterior algebra structure [17].

Accordingly, we have the direct system (4) of the structure algebras $\mathcal{O}^*_k = \Gamma(J^kY, \mathfrak{O}^*_k)$ of global sections of sheaves $\mathfrak{O}^*_k$, i.e., $\mathcal{O}^*_k$ are differential $\mathbb{R}$-algebras of (global) exterior forms on finite-order jet manifolds J^kY. As was mentioned above, the direct limit of (4) is a differential exterior $\mathbb{R}$-algebra $(\mathcal{O}^*_\infty, \pi^{\infty *}_k)$, isomorphic to the algebra of all exterior forms on finite-order jet manifolds modulo the pull-back identification.

The crucial point is that the limit $\mathcal{O}^*_\infty$ of the direct system (4) of structure algebras of sheaves $\mathfrak{O}^*_k$ fails to coincide with the structure algebra $\mathcal{Q}^*_\infty = \Gamma(J^\infty Y, \mathfrak{Q}^*_\infty)$ of the limit $\mathfrak{Q}^*_\infty$ of the direct system (7) of these sheaves. The sheaf $\mathfrak{Q}^*_\infty$, by definition, is the sheaf of germs of local exterior forms on finite-order jet manifolds. These local forms constitute a presheaf $\mathfrak{O}^*_\infty$ from which the sheaf $\mathfrak{Q}^*_\infty$ is constructed. It means that, given a section $\phi \in \Gamma(\mathfrak{Q}^*_\infty)$ of $\mathfrak{Q}^*_\infty$ over an open subset $U \in J^\infty Y$ and any point $q \in U$, there exists a neighbourhood U_q of q such that $\phi|_{U_q}$ is the pull-back of a local exterior form on some finite-order jet manifold. However, $\mathfrak{O}^*_\infty$ does not coincide with the canonical presheaf $\Gamma(\mathfrak{Q}^*_\infty)$ of sections of the sheaf $\mathfrak{Q}^*_\infty$.

In particular, the $\mathbb{R}$-ring $\mathcal{Q}^0_\infty$ is isomorphic to the above mentioned ring of real locally pull-back functions on $J^\infty Y$. Indeed, any element of $\mathcal{Q}^0_\infty$ defines obviously such a function

on $J^\infty Y$. Conversely, the germs of any locally pull-back function f on $J^\infty Y$ belong to the sheaf $\mathfrak{Q}^*_\infty$, i.e., f is a section of $\mathfrak{Q}^*_\infty$, and different such functions f and f' are different sections of $\mathfrak{Q}^*_\infty$.
There are obvious monomorphisms of algebras $\mathcal{O}^*_\infty \to \mathcal{Q}^*_\infty$ and presheaves $\mathfrak{O}^*_\infty \to \Gamma(\mathfrak{Q}^*_\infty)$. For short, we agree to call $\mathfrak{Q}^*_\infty$ (resp. $\mathcal{Q}^*_\infty$) the sheaf (resp. algebra) of locally pull-back exterior forms on $J^\infty Y$. The exterior algebra operations and differentials d, d_V, d_H are defined on $\mathcal{Q}^*_\infty$ just as on $\mathcal{O}^*_\infty$. At the same time, it should be emphasized again that elements of the differential algebras $\mathcal{O}^*_\infty$ and $\mathcal{Q}^*_\infty$ are not differential forms on $J^\infty Y$ in a rigorous sense. Therefore, the standard theorems, e.g., the well-known De Rham theorem ([15], Theorem 2.12.3) can not be applied automatically to these differential algebras.

3 De Rham cohomology

Let $Y \to X$ be an arbitrary smooth fibre bundle. We consider the complex of sheaves of $\mathcal{Q}^0_\infty$-modules

$$0 \to \mathbb{R} \to \mathfrak{Q}^0_\infty \xrightarrow{d} \mathfrak{Q}^1_\infty \xrightarrow{d} \cdots \tag{10}$$

on the infinite-order jet space $J^\infty Y$ and the corresponding infinite-order De Rham complex

$$0 \to \mathbb{R} \to \mathcal{Q}^0_\infty \xrightarrow{d} \mathcal{Q}^1_\infty \xrightarrow{d} \cdots \tag{11}$$

of locally pull-back exterior forms on $J^\infty Y$.
Since locally pull-back exterior forms fulfill the Poincaré lemma, the complex of sheaves (10) is exact. Since the paracompact space $J^\infty Y$ admits a partition of unity performed by elements of $\mathcal{Q}^0_\infty$ [5], the sheaves $\mathfrak{Q}^r_\infty$ of $\mathcal{Q}^0_\infty$-modules are fine for all $r \geq 0$ [4, 15]. Then they are acyclic, i.e., the cohomology groups $H^{>0}(J^\infty Y, \mathfrak{Q}^r_\infty)$ of the paracompact space $J^\infty Y$ with coefficients in the sheaf $\mathfrak{Q}^r_\infty$ vanish [15]. Consequently, the exact sequence (10) is a fine resolution of the constant sheaf $\mathbb{R}$ of germs of local constant real functions on $J^\infty Y$. Then the well-known (generalized De Rham) theorem on a resolution of a sheaf on a paracompact space ([15], Theorem 2.12.1) can be called into play in order to find the cohomology groups of the infinite-order De Rham complex complex (6).
In accordance with this theorem, we have an isomorphism

$$H^*(\mathcal{Q}^*_\infty) = H^*(J^\infty Y, \mathbb{R}) \tag{12}$$

of the De Rham cohomology groups $H^*(\mathcal{Q}^*_\infty)$ of the differential algebra $\mathcal{Q}^*_\infty$ and the cohomology groups $H^*(J^\infty Y, \mathbb{R})$ of the infinite-order jet space $J^\infty Y$ with coefficients in the constant sheaf $\mathbb{R}$. In the case of a vector bundle $Y \to X$, we can say something more.

LEMMA 1. *If $Y \to X$ is a vector bundle, there is an isomorphism*

$$H^*(J^\infty Y, \mathbb{R}) = H^*(X, \mathbb{R}) \tag{13}$$

of cohomology groups of the infinite-order jet space $J^\infty Y$ with coefficients in the constant sheaf $\mathbb{R}$ and those $H^*(X, \mathbb{R})$ of the base X.

Proof. The cohomology groups with coefficient in the constant sheaf $\mathbb{R}$ on homotopic paracompact topological spaces are isomorphic [18]. If $Y \to X$ is a vector bundle, its base X is a strong deformation retract of the infinite-order jet space $J^\infty Y$. To show this, let us consider the map

$$[0,1] \times J^\infty Y \ni (t; x^\lambda, y^i_\Lambda) \to (x^\lambda, t y^i_\Lambda) \in J^\infty Y.$$

A glance at the transition functions (3) shows that, given in the coordinate form, this map is well-defined if $Y \to X$ is a vector bundle. It is a desired homotopy from $J^\infty Y$ to the base X which is identified with its image under the global zero section of the vector bundle $J^\infty Y \to X$. □

Combining the isomorphisms of cohomology groups (12), (13), and the well-known isomorphism $H^*(X, \mathbb{R}) = H^*(X)$, we come to the manifested isomorphism

$$H^*(\mathcal{Q}^*_\infty) = H^*(X)$$

of the De Rham cohomology groups $H^*(\mathcal{Q}^*_\infty)$ of the differential algebra $\mathcal{Q}^*_\infty$ of locally pull-back forms on the infinite-order jet space $J^\infty Y$ of a vector bundle $Y \to X$ to the De Rham cohomology groups $H^*(X)$ of the base X.

It follows that any closed form $\phi \in \mathcal{Q}^*_\infty$ on $J^\infty Y$ is decomposed into the sum $\phi = \varphi + d\xi$ where $\varphi \in \mathcal{O}^*(X)$ is a closed form on X.

4 Cohomology of d_V

Due to the nilpotency rule (5), the vertical and horizontal differentials d_V and d_H on the differential exterior algebra $\mathcal{Q}^*_\infty$ define the bicomplex

$$
\begin{array}{ccccccccc}
 & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & \\
0\to & \mathcal{O}^{k,0}_\infty & \xrightarrow{d_H} & \mathcal{Q}^{k,1}_\infty & \xrightarrow{d_H}\cdots & \mathcal{Q}^{k,m}_\infty & \xrightarrow{d_H}\cdots & \mathcal{Q}^{k,n}_\infty & \\
 & \vdots & & \vdots & & \vdots & & \vdots & \\
 & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & \\
0\to\mathbb{R}\to & \mathcal{Q}^0_\infty & \xrightarrow{d_H} & \mathcal{Q}^{0,1}_\infty & \xrightarrow{d_H}\cdots & \mathcal{Q}^{0,m}_\infty & \xrightarrow{d_H}\cdots & \mathcal{Q}^{0,n}_\infty & \\
 & {\scriptstyle \pi^{\infty*}}\uparrow & & {\scriptstyle \pi^{\infty*}}\uparrow & & {\scriptstyle \pi^{\infty*}}\uparrow & & {\scriptstyle \pi^{\infty*}}\uparrow & \\
0\to\mathbb{R}\to & \mathcal{Q}^0(X) & \xrightarrow{d} & \mathcal{Q}^1(X) & \xrightarrow{d}\cdots & \mathcal{Q}^m(X) & \xrightarrow{d}\cdots & \mathcal{Q}^n(X) & \xrightarrow{d} 0 \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
 & 0 & & 0 & & 0 & & 0 &
\end{array}
\tag{14}
$$

[5-8,10,11,13,14]. The rows and columns of these bicomplex are horizontal and vertical complexes. Let us consider a vertical one

$$0 \to \mathcal{Q}^m(X) \xrightarrow{\pi^{\infty*}} \mathcal{Q}^{0,m}_\infty \xrightarrow{d_V} \cdots \xrightarrow{d_V} \mathcal{Q}^{k,m}_\infty \xrightarrow{d_V} \cdots, \qquad m \leq n. \tag{15}$$

PROPOSITION 2. *If $Y \to X$ is a vector bundle, then the vertical complex (15) is exact.*

Proof. Local exactness of a vertical complex on a coordinate chart $((\pi^\infty)^{-1}(U_X); x^\lambda, y^i_\Lambda)$, $0 \leq |\Lambda|$, on $J^\infty Y$ follows from a version of the Poincaré lemma with parameters (see, e.g., [13]). We have the the corresponding homotopy operator

$$\sigma = \int_0^1 t^k [\overline{y}]\phi(x^\lambda, ty^i_\lambda)]dt, \qquad \phi \in \mathcal{Q}^{k,m}_\infty,$$

where $\overline{y} = y^i_\Lambda \partial^\Lambda_i$. Since $Y \to X$ is a vector bundle, it is readily observed that this homotopy operator is globally defined on $J^\infty Y$, and so is the exterior form σ. $\Box$

It means that any d_V-closed form $\phi \in \mathcal{Q}^*_\infty$ is the sum $\phi = \varphi + d_V\xi$ of a d_V-exact form and an exterior form φ on X.

Of course, d_V-cohomology of the bicomplex (14) is not trivial if the typical fibre of the fibre bundle $Y \to X$ is not contractible.

5 Cohomology of d_H

Turn now to the rows of the bicomplex (14) (excluding the bottom one which is obviously the De Rham complex on the base X). The algebraic Poincaré lemma (see, e.g., [13, 14]) is obviously extended to elements of $\mathcal{Q}^*_\infty$.

PROPOSITION 3. *If $Y \to X$ is a contractible fibre bundle $\mathbb{R}^{n+m} \to \mathbb{R}^n$, the rows of the bicomplex (14) are exact, i.e., they are always locally exact.*

It follows that the the corresponding complexes of sheaves of contact forms

$$0 \to \mathfrak{Q}^{k,0}_\infty \xrightarrow{d_H} \mathfrak{Q}^{k,1}_\infty \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathfrak{Q}^{k,n}_\infty, \qquad k > 0, \tag{16}$$

and the above mentioned horizontal complex

$$0 \to \mathbb{R} \to \mathfrak{Q}^0_\infty \xrightarrow{d_H} \mathfrak{Q}^{0,1}_\infty \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathfrak{Q}^{0,n}_\infty \tag{17}$$

are exact. Recall that, since $J^\infty Y$ is paracompact and admits a partition of unity by elements of $\mathcal{Q}^0_\infty$, all sheaves except the constant sheaf $\mathbb{R}$ in the complexes (16), (17) are fine. However, the exact sequences (16) and (17) fail to be fine resolutions of the sheaves $\mathfrak{Q}^{k,0}_\infty$ and $\mathbb{R}$, respectively, because of their last terms. At the same time, following directly the proof of the above mentioned generalized De Rham theorem ([15], Theorem 2.12.1) till these terms, one can show the following.

PROPOSITION 4. *If $Y \to X$ is an arbitrary smooth fibre bundle, then the cohomology groups $H^r(k, d_H)$, $r < n$, of the complex of $\mathcal{Q}^0_\infty$-modules*

$$0 \to \mathcal{Q}^{k,0}_\infty \xrightarrow{d_H} \mathcal{Q}^{k,1}_\infty \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathcal{Q}^{k,n}_\infty$$

are isomorphic to the cohomology groups $H^r(J^\infty Y, \mathfrak{Q}^{k,0})$ of $J^\infty Y$ with coefficients in the sheaf $\mathfrak{Q}^{k,0}$ and, consequently, are trivial because the sheaf $\mathfrak{Q}^{k,0}$ is fine.

PROPOSITION 5. *If $Y \to X$ is an arbitrary smooth fibre bundle, the cohomology groups $H^r(d_H)$, $r < n$, of the horizontal complex (9) are isomorphic to the cohomology groups $H^r(J^\infty Y, \mathbb{R})$ of $J^\infty Y$ with coefficients in the constant sheaf $\mathbb{R}$.*

Note that one can also study the exact sequence of presheaves

$$0 \to \mathbb{R}_\infty \to \mathfrak{O}^0_\infty \xrightarrow{d} \mathfrak{O}^1_\infty \xrightarrow{d} \cdots,$$

but comes again to the results of Propositions 4, 5. Because $J^\infty Y$ is paracompact, the cohomology groups $H^*(J^\infty Y, \mathfrak{Q}^*_\infty)$ of $J^\infty Y$ with coefficients in the sheaf $\mathfrak{Q}^*_\infty$ and those $H^*(J^\infty Y, \mathfrak{O}^*_\infty)$ with coefficients in the presheaf $\mathfrak{O}^*_\infty$ are isomorphic. It follows that the

cohomology group $H^0(J^\infty Y, \mathfrak{O}^*_\infty)$ of the presheaf $\mathcal{O}^*_\infty$ is isomorphic to the $\mathbb{R}$-module $\mathcal{Q}^*_\infty = H^0(J^\infty Y, \mathfrak{Q}^*_\infty)$, but not $\mathcal{O}^*_\infty$.
If $Y \to X$ is a vector bundle, Lemma 1 and Proposition 5 lead to the manifested isomorphism

$$H^r(d_H) = H^r(X, \mathbb{R}) = H^r(X), \qquad r < n,$$

of the d_H-cohomology groups $H^{<n}(d_H)$ of the horizontal complex (9) to the De Rham cohomology groups $H^{<n}(X)$ of the base X. Then, combining Propositions 4 and 5, we conclude that any d_H-closed form $\phi \in \mathcal{Q}^*_\infty$ on the infinite-order jet manifold $J^\infty Y$ is decomposed into the sum

$$\phi = \varphi + d_H \xi \tag{18}$$

where $\varphi \in \mathcal{O}^*(X)$ is a closed form on X.
Turn to outcomes of this result to BRST theory. Since $\mathbf{s}\varphi = 0$, the decomposition (18) prevents one from the topological obstruction to definition of global descent equations in BRST theory on vector (and affine) bundles.

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